

TIMESTEPPING SCHEMES FOR THE 3D NAVIER–STOKES EQUATIONS: SMALL SOLUTIONS AND SHORT TIMES

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ABSTRACT. It is well known that the solution of the 3d Navier–Stokes equations remains bounded if the initial data and the forcing are sufficiently small relative to the viscosity, and for a finite time given any bounded initial data. In this article, we consider two temporal discretisations (semi-implicit and fully implicit) of the 3d Navier–Stokes equations in a periodic domain and prove that their solutions remain bounded in H^1 subject to essentially the same smallness conditions (on initial data, forcing or time) as the continuous system and to suitable timestep restrictions.

1. INTRODUCTION

Much work has been done on the stability and convergence of various timestepping schemes for the Navier–Stokes equations in two space dimensions (2d NSE). The stability of Euler schemes for 2d NSE has been treated in, e.g., [2, 4, 6, 8], and more recently extended to higher-order schemes in [3, 9]. Given sufficient boundedness of the numerical solution, convergence can usually be established using now-standard techniques (cf., e.g., [5]).

In three dimensions, boundedness of the solution for a finite time (depending on the initial data) follows essentially from the Cauchy–Kovalevskaya theorem. It is also well known that, if both the initial data and the forcing are sufficiently small (relative to the viscosity), the solution will be globally bounded. For more background on the NSE, see, e.g., [1, 7].

In this article we consider temporal discretisations of the 3d NSE using the semi-implicit (2.1) and fully implicit (3.1) schemes, and following ideas from [8] prove discrete analogues of the short-time and small-data boundedness of the continuous-time case. As in the earlier works cited above, we do not consider spatial discretisations, giving the advantage that our results will be free of Courant–Friedrichs–Lewy-type constraints, although some smallness of the timestep may be required.

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We consider the Navier–Stokes equations in $\Omega = (0, 2\pi)^3$ with periodic boundary conditions,

$$(1.1) \quad \begin{aligned} \partial_t u + (u \cdot \nabla)u + \nabla p &= \nu \Delta u + f, \\ \nabla \cdot u &= 0, \end{aligned}$$

plus the initial data $u(0) = u_0$. With no loss of generality, we assume that $\nabla \cdot f = 0$, and that the integrals of f and u_0 vanish over Ω . The last assumption implies that $u = u(t)$, whenever it is well-defined for $t \geq 0$, also has vanishing integral over Ω , giving us the Poincaré inequality

$$(1.2) \quad |u|_{L^2}^2 \leq c_0(\Omega) |\nabla u|_{L^2}^2.$$

For notational convenience, we redefine c_0 to give also the bound

$$(1.3) \quad |\nabla u|_{L^2}^2 \leq c_0 |\Delta u|_{L^2}^2.$$

In order to facilitate comparison with the numerical solutions, in the rest of this section we briefly review the boundedness of solutions of the 3d NSE, both in L^2 and in H^1 for the two cases (small data and short time).

Multiplying (1.1) by u in $L^2(\Omega)$, integrating by parts and using the fact that $(u \cdot \nabla u, u) = 0$, we find

$$(1.4) \quad \frac{1}{2} \frac{d}{dt} |u|^2 + \nu |\nabla u|^2 = (f, u).$$

Here and henceforth, unadorned norm $|\cdot|$ and inner product (\cdot, \cdot) are taken to be L^2 . Bounding the rhs by the Cauchy–Schwarz inequality and using the Poincaré inequality, (1.4) becomes

$$(1.5) \quad \frac{d}{dt} |u|^2 + \frac{\nu}{c_0} |u|^2 \leq \frac{1}{\nu} |f|_{L^\infty(H^{-1})}^2,$$

where $|f|_{L^\infty(H^{-1})} := \sup_{t \geq 0} |f(t)|_{H^{-1}}$. Integrating, we find the uniform L^2 bound

$$(1.6) \quad |u(t)|^2 \leq |u(0)|^2 + (c_0/\nu^2) |f|_{L^\infty(H^{-1})}^2 =: K_0(u_0, f; \nu, \Omega).$$

1.1. H^1 estimate for small solutions. Now multiplying (1.1) by $-\Delta u$ in $L^2(\Omega)$ and integrating by parts, we find

$$(1.7) \quad \frac{1}{2} \frac{d}{dt} |\nabla u|^2 + \nu |\Delta u|^2 = (u \cdot \nabla u, \Delta u) - (f, \Delta u).$$

Bounding the nonlinear term using the Sobolev inequality,

$$(1.8) \quad |(u \cdot \nabla u, \Delta u)| \leq |u|_{L^3} |\nabla u|_{L^6} |\Delta u|_{L^2} \leq \frac{c_1}{2} |u|_{L^3} |\Delta u|_{L^2}^2,$$

and the forcing term in the obvious fashion, we arrive at

$$(1.9) \quad \frac{d}{dt} |\nabla u|^2 + (3\nu/2 - c_1 |u|_{L^3}) |\Delta u|^2 \leq \frac{2}{\nu} |f|^2.$$

Assuming for now that

$$(1.10) \quad |u(t)|_{L^3} \leq \nu/(2c_1) \quad \text{for all } t \geq 0,$$

we can integrate the differential inequality to obtain

$$(1.11) \quad |\nabla u(t)|^2 \leq |\nabla u(0)|^2 + (2c_0/\nu^2)|f|_{L^\infty(L^2)}^2 =: K_1(u_0, f; \nu, \Omega),$$

where $|f|_{L^\infty(L^2)} := \sup_{t \geq 0} |f(t)|_{L^2}$. Using the Sobolev inequality $|u|_{L^3}^2 \leq c|u|_{H^{1/2}}^2 \leq c|u||\nabla u|$, a sufficient condition for (1.10) is

$$(1.12) \quad K_0 K_1 = (|u_0|^2 + c_0 |f|_{L^\infty(H^{-1})}^2) \left(|\nabla u_0|^2 + \frac{2c_0}{\nu^2} |f|_{L^\infty(L^2)}^2 \right) \leq c_2(\Omega) \nu^4.$$

It therefore follows that whenever this holds, the 3d NSE has a global solution bounded by (1.6) and (1.11). It will be convenient to use the Poincaré inequality to derive a slightly stronger condition that implies (1.12),

$$(1.13) \quad K_1 = (|\nabla u_0|^2 + 2c_0 |f|_{L^\infty(L^2)}^2/\nu^2) \leq c_3(\Omega) \nu^2,$$

with $c_3 = \sqrt{c_2/c_0}$.

1.2. H^1 estimate for short times. Multiplying (1.1) by $-\Delta u$ in $L^2(\Omega)$ and integrating by parts, we find

$$(1.14) \quad \frac{1}{2} \frac{d}{dt} |\nabla u|^2 + \nu |\Delta u|^2 = (u \cdot \nabla u, \Delta u) - (f, \Delta u).$$

Bounding the nonlinear term using the Sobolev and interpolation inequalities,

$$(1.15) \quad \begin{aligned} |(u \cdot \nabla u, \Delta u)| &\leq |u|_{L^6} |\nabla u|_{L^3} |\Delta u|_{L^2} \leq c |\nabla u| |\nabla u|_{H^{1/2}} |\Delta u| \\ &\leq c |\nabla u|^{3/2} |\Delta u|^{3/2} \leq \frac{c_4}{2\nu^3} |\nabla u|^6 + \frac{\nu}{2} |\Delta u|^2, \end{aligned}$$

and using the Cauchy–Schwarz inequality for the last term, this gives

$$(1.16) \quad \frac{d}{dt} |\nabla u|^2 \leq \frac{c_4}{\nu^3} |\nabla u|^6 + \frac{1}{\nu} |f|_{L^\infty(L^2)}^2.$$

This implies

$$(1.17) \quad \frac{d}{dt} (|\nabla u|^2 + F) \leq \frac{c_4}{\nu^3} (|\nabla u|^2 + F)^3,$$

where $F := (\nu^2 |f|_{L^\infty(L^2)}^2 / c_4)^{1/3}$. Writing $z(t) := |\nabla u(t)|^2 + F$ and integrating, we find

$$(1.18) \quad \frac{z(t)^2}{z(0)^2} \leq \frac{1}{1 - 2tc_4 z(0)^2 / \nu^3},$$

as long as $t < \nu^3 / (2c_4 z(0)^2)$. It is clear from this that our solution will remain bounded, say, $z(t)^2 \leq 2z(0)^2$, for $0 \leq t \leq \nu^3 / (4c_4 z(0)^2)$.

2. SEMI-IMPLICIT SCHEME

Given a fixed $k > 0$, we discretise (1.1) in time using the semi-implicit Euler scheme

$$(2.1) \quad \frac{u^n - u^{n-1}}{k} + u^{n-1} \cdot \nabla u^n + \nabla p = \nu \Delta u^n + f^n,$$

with $u^0 = u_0$. For 2d NSE, this scheme was proved in [4] to be globally stable in H^1 . For 3d NSE, its stability mirrors that (which is known) of the continuous system, subject to relatively mild timestep restrictions:

Theorem 1. *For small solutions, let the initial data $u_0 \in H^1$, the forcing f and the timestep k satisfy*

$$(2.2) \quad (K_0 + k|f|_{L^\infty(H^{-1})}^2/\nu)(K_1 + 2k|f|_{L^\infty(L^2)}^2/\nu) \leq c_2(\Omega)\nu^4,$$

where $K_0(u_0, f)$ and $K_1(u_0, f)$ are as in the continuous case, (1.6) and (1.11). Then u^n is bounded in H^1 as follows,

$$(2.3) \quad |\nabla u^n|^2 \leq K_1 + (2k/\nu)|f|_{L^\infty(L^2)}^2 \quad \text{for all } n \geq 0.$$

For short times, assuming the timestep restriction (2.20), we have

$$(2.4) \quad |\nabla u^n|^2 \leq 2|\nabla u^0|^2 + F,$$

for all n such that $nk = t_n \leq \nu^3/(8c_4(|\nabla u^0|^2 + F)^2)$.

We note a few facts that will be useful later. First, for any a and $b \in L^2$,

$$(2.5) \quad 2(a - b, a) = |a|^2 - |b|^2 + |a - b|^2.$$

Next, for $b > 0$ and positive sequences (x_n) and (r_n) ,

$$(2.6) \quad (1+b)x_n \leq x_{n-1} + r_{n-1} \quad \Rightarrow \quad x_n \leq (1+b)^{-n}x_0 + \frac{1+b}{b}\max_j r_j.$$

Proof. The L^2 bound works out essentially as in the continuous case: multiplying (2.1)₁ by $2ku^n$, using (2.5) and noting that $(u^{n-1} \cdot \nabla u^n, u^n) = 0$, we find

$$(2.7) \quad |u^n|^2 + |u^n - u^{n-1}|^2 + 2\nu k |\nabla u^n|^2 = |u^{n-1}|^2 + 2k(f^n, u^n).$$

Bounding the forcing term using Cauchy–Schwarz and Poincaré, this implies

$$(2.8) \quad (1 + \nu k/c_0)|u^n|^2 \leq |u^{n-1}|^2 + k|f^n|_{H^{-1}}^2/\nu.$$

Integrating this using (2.6), we find for all $n \in \{1, 2, \dots\}$,

$$(2.9) \quad \begin{aligned} |u^n|^2 &\leq |u^0|^2 + \frac{c_0 + \nu k}{\nu^2} |f|_{L^\infty(H^{-1})}^2 \\ &= K_0(u_0, f; \nu, \Omega) + (k/\nu) |f|_{L^\infty(H^{-1})}^2, \end{aligned}$$

where K_0 , and K_1 below, are as in the continuous case. We note that this bound tends to K_0 as $k \rightarrow 0$.

We now turn to stability in H^1 for small solutions. Multiplying (2.1) by $-2k\Delta u^n$ and using (2.5), we find

$$(2.10) \quad \begin{aligned} & |\nabla u^n|^2 + |\nabla(u^n - u^{n-1})| + 2\nu k |\Delta u^n|^2 \\ &= |\nabla u^{n-1}|^2 - 2k(f^n, \Delta u^n) + 2k(u^{n-1} \cdot \nabla u^n, \Delta u^n). \end{aligned}$$

Bounding the nonlinear term using the Sobolev inequality,

$$(2.11) \quad |(u^{n-1} \cdot \nabla u^n, \Delta u^n)| \leq |u^{n-1}|_{L^3} |\nabla u^n|_{L^6} |\Delta u^n|_{L^2} \leq c_1 |u^{n-1}|_{L^3} |\Delta u^n|_{L^2}^2,$$

and using the Cauchy-Schwarz inequality for the forcing, (2.10) implies

$$(2.12) \quad |\nabla u^n|^2 + (3\nu/2 - c_1 |u^{n-1}|_{L^3}) k |\Delta u^n|^2 \leq |\nabla u^{n-1}|^2 + (2k/\nu) |f^n|^2.$$

If we now assume that

$$(2.13) \quad |u^{n-1}|_{L^3} \leq \nu/(2c_1),$$

we deduce from (2.12)

$$(2.14) \quad (1 + \nu k/c_0) |\nabla u^n|^2 \leq |\nabla u^{n-1}|^2 + 2k |f^n|^2/\nu.$$

As long as (2.13) holds, we can integrate this using (2.6) to get the bound

$$(2.15) \quad \begin{aligned} |\nabla u^n|^2 &\leq |\nabla u^0|^2 + \frac{2(c_0 + \nu k)}{\nu^2} |f|_{L^\infty(L^2)}^2 \\ &\leq K_1(u_0, f; \nu, \Omega) + (2k/\nu) |f|_{L^\infty(L^2)}^2, \end{aligned}$$

which proves (2.3). As in the continuous case, we now use Sobolev and interpolation inequalities to bound

$$(2.16) \quad |u^{n-1}|_{L^3}^2 \leq c |u^{n-1}|_{H^{1/2}}^2 \leq c |u^{n-1}| |\nabla u^{n-1}|.$$

The timestep restriction (2.2) then becomes a sufficient condition for (2.13). More explicitly, since (2.2) holds at $n = 0$, (2.9) and (2.3) imply that it will hold at $n = 1$ and, by induction, for all $n \in \{2, \dots\}$, i.e. the solution of the scheme (2.1) is bounded uniformly in (discrete) time subject to (2.2). Comparing to (1.12), we note that this condition also depends on the timestep k in addition to u_0 and f . This timestep restriction is however relatively mild compared to that for the fully implicit scheme in §3 below.

For short-time H^1 stability, we bound the nonlinear term in (2.10) as in (1.15),

$$(2.17) \quad |(u^{n-1} \cdot \nabla u^n, \Delta u^n)| \leq \frac{\nu}{2} |\Delta u^n|^2 + \frac{c_4}{2\nu^3} |\nabla u^{n-1}|^4 |\nabla u^n|^2,$$

to give

$$(2.18) \quad |\nabla u^n|^2 \leq |\nabla u^{n-1}|^2 + \frac{c_4 k}{\nu^3} |\nabla u^{n-1}|^4 |\nabla u^n|^2 + \frac{k}{\nu} |f|_{L^\infty(L^2)}^2.$$

We rewrite this as

$$(2.19) \quad \begin{aligned} & \left(1 - \frac{c_4 k}{\nu^3} |\nabla u^{n-1}|^4\right) |\nabla u^n|^2 \\ & \leq \left(1 - \frac{c_4 k}{\nu^3} |\nabla u^{n-1}|^4\right) |\nabla u^{n-1}|^2 + \frac{c_4 k}{\nu^3} |\nabla u^{n-1}|^6 + \frac{k}{\nu} |f|_{L^\infty(L^2)}^2. \end{aligned}$$

Since we are interested in short times, we assume that $|\nabla u^{n-1}|^2 \leq 2|\nabla u^0|^2$ for all relevant n and demand that k satisfy

$$(2.20) \quad k \leq \frac{\nu^3}{2c_4(2|\nabla u^0|^2 + F)^2},$$

where $F > 0$ is that in (1.17). This implies that the brackets in (2.19) are $\geq \frac{1}{2}$; we have added the extra F for later use. With this assumption, (2.19) implies

$$(2.21) \quad \frac{|\nabla u^n|^2 - |\nabla u^{n-1}|^2}{k} \leq \frac{2c_4}{\nu^3} |\nabla u^{n-1}|^6 + \frac{2}{\nu} |f|_{L^\infty(L^2)}^2.$$

Unlike its continuous-time analogue (1.16), this difference inequality implies $|\nabla u^n| < \infty$ for all n , although for sufficiently large time nk it (i.e. the bound) grows without bound as $k \rightarrow 0$. This is a well-known pitfall in discretising differential equations in time. To obtain a finite-time bound on $|\nabla u^n|$, we proceed in analogy with (1.17) and define

$$(2.22) \quad z_n := |\nabla u^n|^2 + F.$$

We then get from (2.21)

$$(2.23) \quad \frac{z_n - z_{n-1}}{k} \leq \frac{2c_4}{\nu^3} z_{n-1}^3 =: g(z_{n-1}).$$

Observe that $g(\zeta) > g(\hat{\zeta})$ whenever $\zeta > \hat{\zeta}$, that is, $g \geq 0$ is strictly monotone increasing function.

Now let ζ_n be the positive solution of the difference equation,

$$(2.24) \quad \frac{\zeta_n - \zeta_{n-1}}{k} = g(\zeta_{n-1}),$$

and observe that $\zeta_n \geq 0$ if $\zeta_{n-1} \geq 0$. Denoting $t_n := nk$, we claim that $\zeta_n \leq \zeta(t_n)$ where $\zeta(\cdot)$ is the solution of the differential equation

$$(2.25) \quad \frac{d\zeta}{dt} = g(\zeta) \quad \text{with } \zeta(t_{n-1}) = \zeta_{n-1}.$$

To show this, we first note that $\zeta(t)$ is non-decreasing since $g \geq 0$. Then

$$(2.26) \quad \zeta(t_n) - \zeta(t_{n-1}) = \int_{t_{n-1}}^{t_n} g(\zeta(t)) dt \geq \int_{t_{n-1}}^{t_n} g(\zeta(t_{n-1})) dt = kg(\zeta_{n-1}),$$

proving our claim. By induction, taking $\zeta(0) = \zeta_0 > 0$ instead of the initial data in (2.24), we then have $\zeta_n \leq \zeta(t_n)$ for all $n \in \{1, 2, \dots\}$. Comparing with the continuous case (1.17)–(1.18), we conclude that $\zeta_n \leq \zeta(t_n) \leq 2\zeta(0) = 2\zeta_0$ for $nk = t_n \leq \nu^3/(8c_4\zeta_0^2)$.

Taking $\zeta_0 = z_0$, clearly $z_n \leq \zeta_n$ for all $n \geq 0$. We therefore have

$$(2.27) \quad |\nabla u^n|^2 \leq 2|\nabla u^0|^2 + F,$$

for all n such that $nk = t_n \leq \nu^3/(8c_4(|\nabla u^0|^2 + F)^2)$, which is half as long as the bound in the continuous case. \square

3. FULLY IMPLICIT SCHEME

We now consider the fully implicit Euler scheme

$$(3.1) \quad \frac{u^n - u^{n-1}}{k} + u^n \cdot \nabla u^n + \nabla p = \nu \Delta u^n + f^n,$$

with $\nabla \cdot u^n = 0$ for all n and $u^0 = u_0$. In two space dimensions, uniform boundedness in H^1 for this scheme was proved in [8], whose ideas we borrow below.

The L^2 bound obtains as before: multiplying (3.1)₁ by $2ku^n$ and using (2.5),

$$(3.2) \quad |u^n|^2 + |u^n - u^{n-1}|^2 + 2\nu k |\nabla u^n|^2 = |u^{n-1}|^2 + 2k(f^n, u^n).$$

Bounding the forcing term in the obvious manner and using Poincaré, this implies

$$(3.3) \quad (1 + \nu k/c_0)|u^n|^2 \leq |u^{n-1}|^2 + k|f^n|_{H^{-1}}^2/\nu.$$

Integrating this using (2.6), we find for all $n \in \{1, 2, \dots\}$,

$$(3.4) \quad \begin{aligned} |u^n|^2 &\leq |u^0|^2 + \frac{c_0 + \nu k}{\nu^2} |f|_{L^\infty(H^{-1})}^2 \\ &= K_0(u_0, f; \nu, \Omega) + (k/\nu) |f|_{L^\infty(H^{-1})}^2. \end{aligned}$$

As before, this bound tends to K_0 as $k \rightarrow 0$. For later use, we define

$$(3.5) \quad \tilde{K}_0(u_0, f; \nu, \Omega) := |u_0|^2 + \frac{2c_0}{\nu^2} |f|_{L^\infty(H^{-1})}^2.$$

The central ingredient for our main results is the following local-in-time estimate:

Lemma 1. *We assume the L^2 uniform bound (3.4) and that $u^{n-1} \in H^1$. Assuming further the timestep restrictions*

$$(3.6) \quad K^{(n-1)} \leq \frac{1}{2} \left(\frac{\nu^3}{3c_4 k} \right)^{1/2},$$

$$(3.7) \quad \left(1 + \frac{c_5}{\nu^4} \tilde{K}_0 K^{(n-1)} \right) K^{(n-1)} + |f|_{L^\infty(H^{-1})}^2/\nu^2 \leq \left(\frac{\nu^3}{12c_4 k} \right)^{1/2},$$

where $K^{(n-1)} := |\nabla u^{n-1}|^2 + (10c_0/\nu) |f|_{L^\infty(L^2)}^2$, then the solution u^n of (3.1) is bounded as $|\nabla u^n|^2 \leq y_1$ where y_1 is the smallest positive root of the cubic equation (3.13).

The crucial point which is not immediately obvious is that $y_1 = |\nabla u^{n-1}|^2 + O(k)$ for small k . By estimating the $O(k)$ more carefully, we obtain our main results:

Theorem 2. *For short times, let the timestep k satisfy (3.29), (3.30) and (3.31) below. Then there exists a t_f^* such that, as long as $0 \leq nk \leq t_f^*$ we have*

$$(3.8) \quad |\nabla u^n|^2 \leq 2|\nabla u_0|^2 + (\nu^2 |f|_{L^\infty(L^2)}^2/c_4)^{1/3}.$$

For small solutions, let u_0 and f be such that

$$(3.9) \quad |\nabla u_0|^2 + \frac{2c_0}{\nu^2} |f|_{L^\infty(L^2)}^2 \leq \frac{\nu^2}{2\sqrt{c_0 c_4}},$$

and let the timestep k satisfy (3.36)–(3.38) below. Then u^n is bounded as

$$(3.10) \quad |\nabla u^n|^2 \leq \tilde{K}_1(u_0, f; \nu, \Omega) := |\nabla u_0|^2 + \frac{10c_0}{\nu^2} |f|_{L^\infty(L^2)}^2,$$

for all $n \in \{0, 1, \dots\}$.

We note that, up to constants depending only on the domain Ω , both the smallness condition (3.9) and the bound (3.10) are the same as those in the continuous case, (1.13) and (1.11). Also, the time bound t_f^* is essentially that in the continuous case (smaller by a factor of $\frac{1}{2}$ which can be improved to $1 - \varepsilon$ with some work and more restriction on k).

Proof of Lemma 1. Multiplying (3.1) by $-2k\Delta u^n$, we have

$$(3.11) \quad \begin{aligned} & |\nabla u^n|^2 + |\nabla(u^n - u^{n-1})|^2 + 2\nu k |\Delta u^n|^2 \\ &= |\nabla u^{n-1}|^2 + 2k(u^n \cdot \nabla u^n, \Delta u^n) - 2k(f^n, \Delta u^n). \end{aligned}$$

Bounding the nonlinear term as we did in (1.15),

$$(3.12) \quad 2k|(u^n \cdot \nabla u^n, \Delta u^n)| \leq \frac{c_4 k}{\nu^3} |\nabla u^n|^6 + \nu k |\Delta u^n|^2,$$

we find

$$(3.13) \quad \begin{aligned} 0 &\leq \frac{c_4 k}{\nu^3} |\nabla u^n|^6 - |\nabla u^n|^2 - \frac{\nu k}{2} |\Delta u^n|^2 + |\nabla u^{n-1}|^2 + \frac{2k}{\nu} |f^n|^2 \\ \Rightarrow \quad 0 &\leq \frac{c_4 k}{\nu^3} |\nabla u^n|^6 - \left(1 + \frac{\nu k}{2c_0}\right) |\nabla u^n|^2 + |\nabla u^{n-1}|^2 + \frac{2k}{\nu} |f|_{L^\infty(L^2)}^2. \end{aligned}$$

Let $y := |\nabla u^n|^2$, $x := |\nabla u^{n-1}|^2 + 2k|f|_{L^\infty(L^2)}^2/\nu$ and

$$(3.14) \quad G(y; x) := (c_4 k / \nu^3) y^3 - (1 + \nu k / (2c_0)) y + x.$$

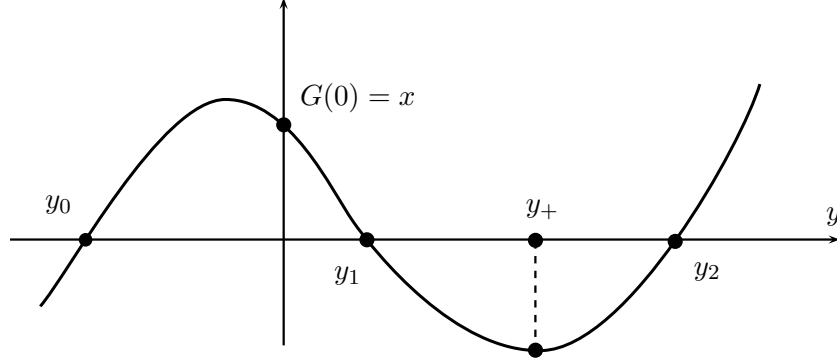
We write $G(y)$ instead of $G(y; x)$ when there is no risk of confusion. We are of course interested in the solution set of $G(y) \geq 0$.

Under the assumption (3.17) below on the timestep k , the graph of the cubic G is (qualitatively) as shown in Figure 1. We note in particular that $G(y) = 0$ has a negative root y_0 and two positive roots y_1 and y_2 . To verify this, we note the following. First, $G(y) \rightarrow \pm\infty$ as $y \rightarrow \pm\infty$. Next, $G(y)$ has two local extrema,

$$(3.15) \quad y_{\pm} = \pm \left(\frac{\nu^3}{3c_4 k} \left[1 + \frac{\nu k}{2c_0} \right] \right)^{1/2},$$

with $y_- < 0$ being a local maximum and $y_+ > 0$ a local minimum, as verified by computing $G''(y_{\pm})$. Since $G(0) = x > 0$ (the problem is trivial if $x = 0$), we have $G(y_-) > 0$. Finally, computing

$$(3.16) \quad G(y_+) = -\frac{2}{3} \left(1 + \frac{\nu k}{2c_0} \right) \left(\frac{\nu^3}{3c_4 k} \left[1 + \frac{\nu k}{2c_0} \right] \right)^{1/2} + x,$$

FIGURE 1. The graph of $G(y)$ in (3.14): y_+ is a local minimum.

we conclude that $G(y_+) < 0$ if (this is essentially a restriction on k)

$$(3.17) \quad |\nabla u^{n-1}|^2 + \frac{2k}{\nu} |f|_{L^\infty(L^2)}^2 < \frac{2}{3} \left(\frac{\nu^3}{3c_4 k} \right)^{1/2}.$$

This implies the existence of the two positive roots y_1 and y_2 with $y_1 < y_+ < y_2$.

Now (3.13) implies that $|\nabla u^n|^2 = y$ lies in the disjoint set $[0, y_1] \cup [y_2, \infty)$. However, $y_2 > y_+ \sim k^{-1/2}$, which is absurd for small k . To prove that $y \notin [y_2, \infty)$, we multiply (3.1)₁ by $2k(u^n - u^{n-1})$ in L^2 to get

$$(3.18) \quad \begin{aligned} 2|u^n - u^{n-1}|^2 + \nu k |\nabla u^n|^2 - \nu k |\nabla u^{n-1}|^2 + \nu k |\nabla(u^n - u^{n-1})|^2 \\ = -2k(u^n \cdot \nabla u^n, u^n - u^{n-1}) + 2k(f^n, u^n - u^{n-1}) =: I_1 + I_2. \end{aligned}$$

Bounding the rhs as

$$\begin{aligned} |I_2| &\leq \frac{k}{\nu} |f^n|_{H^{-1}}^2 + \nu k |\nabla(u^n - u^{n-1})|^2 \\ |I_1| &= 2k |(u^n \cdot \nabla u^n, u^{n-1})| \leq 2k |u^n|_{L^3} |\nabla u^n|_{L^2} |u^{n-1}|_{L^6} \\ &\leq ck |u^n|_{H^{1/2}} |\nabla u^n| |\nabla u^{n-1}| \leq ck |u^n|^{1/2} |\nabla u^n|^{3/2} |\nabla u^{n-1}| \\ &\leq \frac{\nu k}{2} |\nabla u^n|^2 + \frac{ck}{\nu^3} |u^n|^2 |\nabla u^{n-1}|^4, \end{aligned}$$

and dropping the $2|u^n - u^{n-1}|^2$ on the lhs in (3.18), we arrive at

$$(3.19) \quad |\nabla u^n|^2 \leq \left(2 + \frac{2c_5}{\nu^4} |u^n|^2 |\nabla u^{n-1}|^2 \right) |\nabla u^{n-1}|^2 + \frac{2}{\nu^2} |f|_{L^\infty(H^{-1})}^2.$$

If we now assume that (effectively a timestep restriction)

$$(3.20) \quad \left(2 + \frac{2c_5}{\nu^4} |u^n|^2 |\nabla u^{n-1}|^2 \right) |\nabla u^{n-1}|^2 + \frac{2}{\nu^2} |f|_{L^\infty(H^{-1})}^2 \leq \left(\frac{\nu^3}{3c_4 k} \right)^{1/2},$$

noting that the rhs $< y_+ < y_2$, we can conclude that $|\nabla u^n|^2 < y_2$ and therefore $|\nabla u^n|^2 \in [0, y_1]$. This gives us the local H^1 integrability of the

scheme (3.1): if k (is small enough that it) satisfies (3.17) and (3.20), the one-step solution of (3.1) is bounded in H^1 . \square

Proof of Theorem 2. We begin with the short-time case and assume the hypotheses (local in n) of Lemma 1. Since y_1 is the root of a cubic $G(y_1) = 0$, the bound $|\nabla u^n|^2 \leq y_1$ is not very convenient, so we compute a more useful bound. Recalling that $x > 0$, we consider for some $a > 0$

$$(3.21) \quad \begin{aligned} G((1+ak)x; x) &= xk \left[\frac{c_4}{\nu^3} (1+ak)^3 x^2 - \left(\frac{\nu}{2c_0} + a \right) - \frac{a\nu k}{2c_0} \right] \\ &< xk \left[\frac{c_4}{\nu^3} (1+ak)^3 x^2 - a \right]. \end{aligned}$$

Setting $a = 2c_4 x^2 / \nu^3$, this implies $G((1+ak)x) < 0$ if

$$(3.22) \quad 1+ak \leq 2^{1/3} \quad \Leftrightarrow \quad (|\nabla u^{n-1}|^2 + 2k|f|_{L^\infty(L^2)}^2 / \nu)^2 2c_4 k / \nu^3 \leq 2^{1/3} - 1.$$

Assuming this, Lemma 1 then gives us the explicit one-step estimate

$$(3.23) \quad \begin{aligned} |\nabla u^n|^2 &\leq y_1 \leq (1+ak)x \\ &= |\nabla u^{n-1}|^2 + \frac{2k}{\nu} |f|_{L^\infty(L^2)}^2 + \frac{2c_4 k}{\nu^3} (|\nabla u^{n-1}|^2 + (2k/\nu) |f|_{L^\infty(L^2)}^2)^3, \end{aligned}$$

which we can rewrite as

$$(3.24) \quad \frac{|\nabla u^n|^2 - |\nabla u^{n-1}|^2}{k} \leq \frac{2c_4}{\nu^3} \left[\left(|\nabla u^{n-1}|^2 + \frac{2k}{\nu} |f|_{L^\infty(L^2)}^2 \right)^3 + \frac{\nu^2}{c_4} |f|_{L^\infty(L^2)}^2 \right].$$

To obtain a finite-time bound on $|\nabla u^n|$, we proceed in analogy with (1.17) and define

$$(3.25) \quad z_n := |\nabla u^n|^2 + F \quad \text{where } F^3 = \frac{2\nu^2}{c_4} |f|_{L^\infty(L^2)}^2.$$

By expanding both sides, we have

$$(3.26) \quad \left(|\nabla u^{n-1}|^2 + \frac{2k}{\nu} |f|_{L^\infty(L^2)}^2 \right)^3 + \frac{\nu^2}{c_4} |f|_{L^\infty(L^2)}^2 \leq z_{n-1}^3,$$

subject to the timestep restriction

$$(3.27) \quad k \leq \frac{\nu^{5/3}}{2c_4^{1/3} |f|_{L^\infty(L^2)}^{4/3}} \quad \Rightarrow \quad \begin{cases} 4^{1/3} c_4^{1/3} |f|_{L^\infty(L^2)}^{4/3} k \leq \nu^{5/3}, \\ 4^{2/3} c_4^{2/3} |f|_{L^\infty(L^2)}^{8/3} k^2 \leq \nu^{10/3}, \\ 8c_4 |f|_{L^\infty(L^2)}^4 k^3 \leq \nu^5. \end{cases}$$

Then (3.23) implies

$$(3.28) \quad \frac{z_n - z_{n-1}}{k} \leq \frac{2c_4}{\nu^3} z_{n-1}^3 =: g(z_{n-1}).$$

Arguing as we did in the semi-implicit case [cf. (2.24)–(2.26)], we conclude that $z_n \leq 2z_0$ for all $n \geq 0$ such that $nk = t_n \leq \nu^3 / (8c_4 \zeta_0^2)$.

This proves the theorem subject to the timestep restrictions, which we collect here. First, (3.6) and (3.7) are implied by

$$(3.29) \quad \tilde{K} \leq \frac{1}{2} \left(\frac{\nu^3}{3c_4 k} \right)^{1/2},$$

$$(3.30) \quad \left(1 + \frac{c_5}{\nu^4} \tilde{K}_0 \tilde{K} \right) \tilde{K} + |f|_{L^\infty(H^{-1})}^2 / \nu^2 \leq \left(\frac{\nu^3}{12c_4 k} \right)^{1/2},$$

where unlike in Lemma 1, here $\tilde{K} := 2|\nabla u^0|^2 + 2(\nu^2 |f|_{L^\infty(L^2)}^2 / c_4)^{1/3} + (10c_0/\nu) |f|_{L^\infty(L^2)}^2$. Next, (3.27) is good as it stands. Finally, using (3.27) to handle the k inside the bracket, (3.22) is implied by

$$(3.31) \quad \left(2|\nabla u^0|^2 + \frac{(1 + 2^{1/3})\nu^{2/3} |f|_{L^\infty(L^2)}^{2/3}}{c_4^{1/3}} \right)^2 \leq \frac{(2^{1/3} - 1)\nu^3}{2c_4 k}.$$

This proves the short-time case.

For small solutions, we first derive a more useful explicit bound for $|\nabla u^{n-1}|^2$. We claim that with the assumption (3.9), $|\nabla u^n|^2 \leq y_1$ implies

$$(3.32) \quad \left(1 + \frac{\nu k}{4c_0} \right) |\nabla u^n|^2 \leq |\nabla u^{n-1}|^2 + \frac{2k}{\nu} |f|_{L^\infty(L^2)}^2.$$

To prove this, we set $y_* := (|\nabla u^{n-1}|^2 + 2k |f|_{L^\infty(L^2)}^2 / \nu) / (1 + \nu k / (4c_0))$ and compute

$$(3.33) \quad G(y_*) = y_* \left(1 + \frac{\nu k}{4c_0} \right)^{-2} \left\{ -\frac{\nu k}{4c_0} \left(1 + \frac{\nu k}{4c_0} \right)^2 + \frac{c_4 k}{\nu^3} x^2 \right\}.$$

Now $G(y_*) \leq 0$ implies that $y_* \geq y_1$, and the former is true if

$$(3.34) \quad |\nabla u^{n-1}|^2 + \frac{2k}{\nu} |f|_{L^\infty(L^2)}^2 = x \leq \frac{\nu^2}{2\sqrt{c_0 c_4}}.$$

To obtain the uniform bound, we sum (3.32) using (2.6) to find

$$(3.35) \quad |\nabla u^n|^2 \leq \left(1 + \frac{\nu k}{4c_0} \right)^{-n} |\nabla u^0|^2 + \frac{8c_0}{\nu^2} |f|_{L^\infty(L^2)}^2 + \frac{2k}{\nu} |f|_{L^\infty(L^2)}^2.$$

Assuming that

$$(3.36) \quad k \leq c_0 / \nu,$$

we can absorb the last term into the penultimate one to obtain (3.10). Consolidating our assumptions, the smallness condition (3.34) is now implied by (3.9), while the timestep restrictions (3.17) and (3.20) can both be satisfied by taking k sufficiently small to satisfy

$$(3.37) \quad \tilde{K}_1 \leq \frac{1}{2} \left(\frac{\nu^3}{3c_4 k} \right)^{1/2},$$

$$(3.38) \quad \left(1 + \frac{c_5}{\nu^4} \tilde{K}_0 \tilde{K}_1 \right) \tilde{K}_1 + |f|_{L^\infty(H^{-1})}^2 / \nu^2 \leq \left(\frac{\nu^3}{12c_4 k} \right)^{1/2}.$$

This proves the theorem. \square

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